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METHOD OF EXTENSION OF BOUNDARIES FOR PROBLEMS OF HEAT CONDUCTION IN BODIES OF MOBILE SHAPE

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The first boundary-value problem of heat conduction was solved by the method of extension of boundaries with the simplest example of a variable-length segment. An analytical solution of the problem was obtained for arbitrary initial conditions and the law of motion of a boundary with the aid of the Fourier rapidly convergent series. An example of the solution of the problem for the case where the left end of the segment is fixed and the right one moves with a constant velocity is given.

Keywords: *mobile shape of the boundary, heat conduction, boundary function, Fourier convergent series.*

Introduction. Various applied problems can be posed for regions with a time-varying shape. These include problems of a biological nature as well as those for elastic and plastic bodies in the Euler variables, for heat conduction and mass transfer in bodies when material is deposited on the body surface by sputtering, or, conversely, part of its mass is removed from the body surface as a result of melting or abrasion of loose material, and in other cases.

Analogous problems with a moving boundary have been little studied mathematically and are the most difficult to solve. Some solutions were obtained for geometrically one-dimensional heat-conduction problems, and only a few solutions are known for two-dimensional problems. A rather detailed statement and review of these works can be found in [1, 2]. Some solutions of multidimensional problems were obtained in [3]. Unique numerical-analytical methods of solving similar problems were considered in [4]. It is suggested to construct the solution of the problems for regions of mobile shape with the aid of a new method — the method of extension of boundaries.

Statement of the Problem and Its Solution. The essence of the method is as follows:

a) the replacement of the considered region Ω by a wider auxiliary one $\Omega_{\square}$ of a certain classical shape, i.e., $\Omega \in \Omega_{\square}$, for which the spectra of eigenfunctions and eigenvalues are known;

b) statement of unknown boundary conditions on a new boundary which are determined when the real boundary conditions are fulfilled;

c) extension of the definition of the initial conditions and internal sources in the extension $\Omega_{\square} \setminus \Omega$ with a certain smoothness (the degree of smoothness influences the rate of convergence of an approximate solution to an exact one).

The problem is posed as follows: it is required to find boundary conditions on the new boundary and the solution of the problem for the extended region $\Omega_{\square}$ such that this solution could satisfy the boundary conditions on the initial boundary.

The simplicity of the method allows one to apply it also to the solution of especially complex problems with phase transformation. To demonstrate the effectiveness of the method of extension of boundaries with the aid of a simple example, we will consider the following geometrically one-dimensional heat-conduction problem.

For a fixed material with a mobile right boundary we write the heat-conduction equation:

$$U_t = aU_{xx} + f(t, x), \quad x \in \Omega = [0, l(t)], \quad l(t) > 0, \quad 0 < t \le t_0;$$
$$U \in C^{(4)}(\Omega), \quad f(t, x) \in C^{(2)}(\Omega), \quad l(t) \in C \quad (0 \le t \le t_0). \tag{1}$$

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The conditions of smoothness in (1) are needed for subsequent use of spectral decompositions [5]. We write the initial and boundary conditions:

$$U(0, x) = \varphi(x)\,, \quad U(t, 0) = \mu_1(t)\,, \quad U(t, l(t)) = \mu_2(t)\,. \tag{2}$$

We introduce a wider fixed region $\Omega_\square$ which contains the mobile region Ω:

$$\Omega_\square = (x \in [0, b])\,, \quad \Omega \in \Omega_\square\,, \quad \text{where} \quad b > l(t)\ \forall\ t \in [0, t_0]\,. \tag{3}$$

Instead of Eqs. (1) and (2), we consider a problem with an immovable boundary:

$$U_t = aU_{xx} + f_\square(t, x)\,, \quad U(0, x) = \varphi_\square(x)\,, \quad x \in \Omega_\square\,, \quad 0 \le t \le t_0\,; \tag{4}$$

$$\mu_1(t) = U(t, 0)\,, \quad \mu_2(t) = U(t, l(t))\,, \quad \mu_3(t) = U(t, b)\,. \tag{5}$$

We extend the functions $f_\square(t, x)$ and $\varphi_\square(x)$ from Ω into the expanded part of the region $\Omega_\square$ by means of selecting artificially parameters complying with the smoothness conditions:

$$f_\square(t, x) = \left\{ \begin{matrix} f(t, x)\,, & \text{if} \quad x \in [0, l(t)] \\ f^*(t, x)\,, & \text{if} \quad x \in [l(t), b] \end{matrix} \right\} \in C^{(2)}(\Omega_\square)\,;$$

$$\varphi_\square(t, x) = \left\{ \begin{matrix} \varphi(x)\,, & \text{if} \quad x \in [0, l(0)] \\ \varphi^*(x)\,, & \text{if} \quad x \in [l(0), b] \end{matrix} \right\} \in C^{(4)}(\Omega_\square)\,. \tag{6}$$

In Eq. (4) an additional unknown function $\mu_3(t)$ is introduced, and therefore problem (4)–(6) is posed as follows: such a solution of the heat-conduction equation is to be found that would satisfy the initial and boundary conditions indicated in (4)–(6). In actual fact, problem (1), (2) with a mobile boundary has been reduced to problem (4)–(6) with an immovable boundary $\Gamma_\square$ for a wider region $\Omega_\square$ with the unknown boundary condition $\mu_3(t)$ on the right boundary $\Gamma_\square$, which will be found from additional condition (5). When $0 \le x \le l(t)$, the statement of problems (1), (2) and (4)–(6) are identical due to the fulfillment of the equalities $f_\square(t, x) = f(t, x)$, $\varphi_\square(x) = \varphi(x)$ at $0 \le x \le l(t)$ by definition. Then, according to the uniqueness theorem, the solutions of these problems at $0 \le x \le l(t)$ will be equal.

We represent the solution of problem (4)–(6) with the aid of the fast Fourier series:

$$U(t, x) = \mu_1(t) + \frac{x}{b}[\mu_3(t) - \mu_1(t)] + \frac{1}{2}x(x - b)\,\mu_4(t)$$

$$+ \frac{x}{6b}(x^2 - b^2)[\mu_5(t) - \mu_4(t)] + \sum_{m=1}^{\infty} U_m(t) \sin m\pi \frac{x}{b}\,, \tag{7}$$

$$\mu_4(t) = U_{xx}(t, 0)\,, \quad \mu_5(t) = U_{xx}(t, b)\,.$$

Here the functions depending only on time are unknown:

$$\mu_3(t)\,, \quad \mu_4(t)\,, \quad \mu_5(t)\,, \quad U_m(t)\,, \quad m = 1, 2, \ldots\,. \tag{8}$$

The construction of the fast Fourier series in (7) admits termwise calculation of partial derivatives of $U(t, x)$ with respect to x up to the fourth order inclusive [6], with the partial derivatives of the first and second order converging uniformly not only inside $\Omega_\square$, but also at the ends of the fixed segment $\Gamma_\square$. Moreover, here the Fourier coefficients $U_m(t)$ have order of decrease with increasing number m not lower than $1/(\pi m)^5$, ensuring a rapid uniform convergence of the series. The internal source $f_{\square}(t, x)$ will be represented identically in the form of a fast Fourier series that converges uniformly inside $\Gamma_{\square}$ and at its ends:

$$f_{\square}(t,x) = f_1^*(t) + \frac{x}{b}(f_2^*(t) - f_1^*(t)) + \sum_{m=1}^{\infty} f_m(t) \sin m\pi \frac{x}{b}, \quad f_1^*(t) = f_{\square}(t, 0),$$

$$f_2^*(t) = f_{\square}(t, b), \quad f_m(t) = \frac{2}{b}\int_0^b f_{\square}(t, x) \sin m\pi \frac{x}{b}\, dx + \frac{2}{m\pi}\left[f_2^*(t)(-1)^m - f_1^*(t)\right]. \tag{9}$$

The properties of the series from Eq. (7) allow one to substitute $U(t, x)$ into differential equation (4) using termwise differentiation:

$$\mu_1' + \frac{x}{b}(\mu_3' - \mu_1') + \frac{1}{2}x(x-b)\mu_4' + \frac{x}{6b}(x^2 - b^2)(\mu_5' - \mu_4')$$

$$+ \sum_{m=1}^{\infty} U_m' \sin m\pi \frac{x}{b} = a\mu_4 + a\frac{x}{b}(\mu_5 - \mu_4) - a\frac{\pi^2}{b^2}\sum_{m=1}^{\infty} m^2 U_m \sin m\pi \frac{x}{b}$$

$$+ f_1^* + \frac{x}{b}(f_2^* - f_1^*) + \sum_{m=1}^{\infty} f_m \sin m\pi \frac{x}{b}. \tag{10}$$

By construction, the series in (10) converge at $x = 0$ and $x = b$; therefore, from this we will have two equations for $\mu_4(t)$ and $\mu_5(t)$:

$$\mu_1' = a\mu_4 + f_1^*, \quad \mu_3' = a\mu_5 + f_2^*. \tag{11}$$

With the aid of expressions (11) equality (10) is simplified significantly:

$$\frac{1}{2}x(x-b)\mu_4' + \frac{x}{6b}(x^2 - b^2)(\mu_5' - \mu_4') + \sum_{m=1}^{\infty}\left(U_m' + a\frac{\pi^2}{b^2}m^2 U_m - f_m\right)\sin m\pi \frac{x}{b} = 0. \tag{12}$$

Assuming in (7) that $x = l(t)$ and substituting $U(t, l)$ into the second boundary condition (5), we arrive at the equation for finding μ_3:

$$\mu_1 + \frac{l}{b}(\mu_3 - \mu_1) + \frac{1}{2}l(l-b)\mu_4 + \frac{l}{6b}(l^2 - b^2)(\mu_5 - \mu_4) + \sum_{m=1}^{\infty} U_m \sin m\pi \frac{l}{b} = \mu_2. \tag{13}$$

The series in (13) converges, since $l < b$. The remaining equations for finding the unknown quantities indicated in (8) will be obtained by expanding the left and right sides of equality (12) with respect to the sines $\sin(m\pi x/b)$ over the segment $[0, b]$:

$$-\frac{2b^2}{m^3\pi^3}\mu_4' + \frac{2b^2}{m^3\pi^3}(-1)^m \mu_5' + U_m' + a\frac{\pi^2}{b^2}m^2 U_m - f_m = 0, \quad m = 1, 2, \ldots. \tag{14}$$

After elimination of μ_4 and μ_5 from (13) and (14) with the aid of (11), we arrive at a closed inhomogeneous linear system of ordinary differential equations for $\mu_3(t)$ and $U_m(t)$:

$$\mu_1+\frac{l}{b}(\mu_3-\mu_1)+\frac{1}{2a}\,l\,(l-b)\,(\mu_1'-f_1^*)+\frac{l}{6ab}(l^2-b^2)\,(\mu_3'-\mu_1'+f_1^*-f_2^*)+\sum_{m=1}^{\infty}U_m\sin m\pi\frac{l}{b}=\mu_2\,;$$

$$U_m'+a\frac{\pi^2}{b^2}m^2U_m=\frac{2b^2}{am^3\pi^3}\left(\mu_1''-f_1^{*\prime}\right)-\frac{2b^2}{am^3\pi^3}(-1)^m\left(\mu_3''-f_2^{*\prime}\right)+f_m\,,\quad m=1,2,\ldots\,. \tag{15}$$

The initial conditions for system (15) will be found from initial conditions (4)–(6), (11) at $t = 0$:

$$\mu_3(0)=\varphi_\square(b)\,,\quad \mu_3'(0)=a\varphi_{\square xx}(b)+f(0,b)\,,$$

$$U_m(0)=\frac{2}{b}\int_0^b\varphi_\square(x)\sin m\pi\frac{x}{b}\,dx-\frac{2}{m\pi}\mu_1(0)+2\frac{(-1)^m}{m\pi}\mu_3(0)+\frac{2b^2}{m^3\pi^3}\mu_4(0)-\frac{2b^2}{m^3\pi^3}(-1)^m\mu_5(0)\,,\quad m=1,2\ldots\,. \tag{16}$$

The case where $l(0) = 0$, i.e., where the process of sputtering of a film of some material is considered from the beginning, is special for system (15), since in the first equation of the system the coefficient of $\mu_3'(t)$ vanishes. Therefore, in a similar case one should use a different condition: $l(0) = \delta$, where δ is a small value which is assumed to be smaller than the admissible error of calculations.

The system of differential conditions (15) with initial conditions (16) can be solved with the aid of one of the well-known standard computer programs. This system has a singularity, which is revealed on its reduction to the standard form $y' = s(t, y)$. For this purpose we will solve the first equation of (15) for μ_3':

$$\mu_3'=\mu_1'-f_1^*+f_2^*+\frac{6ab}{l^3-b^2l}\left(\mu_2-\mu_1+\frac{l}{b}\mu_1-\frac{l^2-lb}{2a}(\mu_1'-f_1^*)\right)$$

$$-\frac{6ab}{l^3-b^2l}\left(U_1\sin\pi\frac{l}{b}+\sum_{p=2}^{\infty}U_p\sin p\pi\frac{l}{b}\right)-\frac{6a}{l^2-b^2}\mu_3\,, \tag{17}$$

Having differentiated Eq. (17) with respect to t and again representing μ_3' with the aid of (17), we find μ_3'' as a function of μ_3, U_m, and U_m':

$$\mu_3''=\mu_1''+f_2^{*\prime}-f_1^{*\prime}-\frac{6ab}{l^3-b^2l}\left(l'\frac{3l^2-b^2}{l^3-b^2l}+\frac{6a}{l^2-b^2}\right)A(t)$$

$$-\frac{6a}{l^2-b^2}\left(\mu_1'-f_1^*+f_2^*\right)+\frac{12a}{(l^2-b^2)^2}(ll'+3a)\,\mu_3$$

$$+\frac{6ab}{l^3-b^2l}\left(C(t)-\sum_{m=1}^{\infty}\left(U_m'\sin m\pi\frac{l}{b}+U_m\,m\pi\frac{l'}{b}\cos m\pi\frac{l}{b}\right)\right)$$

$$+\frac{6ab}{l^3-b^2l}\left(\frac{l'(3l^2-b^2)}{l^3-b^2l}+\frac{6a}{l^2-b^2}\right)\sum_{m=1}^{\infty}U_m\sin m\pi\frac{l}{b}\,, \tag{18}$$

$$A(t)=\mu_2-\mu_1+\frac{l}{b}\mu_1-\frac{l^2-lb}{2a}\left(\mu_1'-f_1^{*\prime}\right);$$

$$C(t) = \mu_2' - \mu_1' + \frac{l'}{b}\mu_1 + \frac{l}{b}\mu_1' - \frac{l^2 - lb}{2a}\left(\mu_1'' - f_1^{*\prime}\right) - \frac{l'(2l-b)}{2a}\left(\mu_1' - f_1^*\right).$$

Now, using (18), we eliminate μ_3'' from Eqs. (15) that contain the subscript m:

$$U_m' + a\frac{\pi^2}{b^2}m^2 U_m = f_m + \frac{2b^2}{am^3\pi^3}\left(\mu_1'' - f_1^{*\prime}\right) + \frac{12b^3(-1)^m}{m^3\pi^3(l^3 - b^2 l)}\sum_{p=1}^{\infty} U_p' \sin p\pi\frac{l}{b}$$

$$-\frac{2b^2}{am^3\pi^3}(-1)^m\left[\mu_1'' - f_1^{*\prime} - \frac{6ab}{l^3 - b^2 l}\left(l'\frac{3l^2 - b^2}{l^3 - b^2 l} + \frac{6a}{l^2 - b^2}\right)A(t)\right.$$

$$+\frac{6ab}{l^3 - b^2 l}\left(C(t) - \sum_{p=1}^{\infty} U_p p\pi\frac{l'}{b}\cos p\pi\frac{l}{b}\right) + \frac{12a}{(l^2 - b^2)^2}(ll' + 3a)\mu_3$$

$$\left.+\frac{6ab}{l^3 - b^2 l}\left(l'\frac{3l^2 - b^2}{l^3 - b^2 l} + \frac{6a}{l^2 - b^2}\right)\sum_{p=1}^{\infty} U_p \sin p\pi\frac{l}{b} - \frac{6a}{l^2 - b^2}\left(\mu_1' - f_1^* + f_2^*\right)\right], \quad m = 1, 2, \ldots . \tag{19}$$

We will resolve linear system (19) for the derivatives U_m'. For this purpose, we multiply the left and right sides by $\sin m\pi l/b$, which is meaningful only when $l < b$, sum up over all the subscripts m, and group the terms with U_m' on the left-hand side of the equality:

$$\left(1 - \sum_{p=1}^{\infty}\frac{12b^3(-1)^p}{p^3\pi^3(l^3 - b^2 l)}\sin p\pi\frac{l}{b}\right)\sum_{p=1}^{\infty} U_p' \sin p\pi\frac{l}{b} = \sum_{p=1}^{\infty} f_p \sin p\pi\frac{l}{b} + \sum_{p=1}^{\infty}\frac{2M_1' b^2}{ap^3\pi^3}\sin p\pi\frac{l}{b}$$

$$-\sum_{p=1}^{\infty}\frac{2b^2}{ap^3\pi^3}(-1)^p \sin p\pi\frac{l}{b}\left[M_1' - \frac{6ab}{l^3 - b^2 l}\left(l'\frac{3l^2 - b^2}{l^3 - b^2 l} + \frac{6a}{l^2 - b^2}\right)A(t)\right.$$

$$-\frac{6a}{l^2 - b^2}\left(M_1 + f_2^*\right) + \frac{6ab}{l^3 - b^2 l}\left(\mu_2' - \mu_1' + \frac{l'}{b}\mu_1 + \frac{l}{b}\mu_1' - \frac{l^2 - lb}{2a}M_1' - l'\frac{2l - b}{2}M_1\right)$$

$$-\frac{6ab}{l^3 - b^2 l}\sum_{p=1}^{\infty} U_p p\pi\frac{l'}{b}\cos p\pi\frac{l}{b} + \frac{12a}{(l^2 - b^2)^2}(ll' + 3a)\mu_3$$

$$\left.+\frac{6ab}{l^3 - b^2 l}\left(l'\frac{3l^2 - b^2}{l^3 - b^2 l} + \frac{6a}{l^2 - b^2}\right)\sum_{p=1}^{\infty} U_p \sin p\pi\frac{l}{b}\right] - a\frac{\pi^2}{b^2}\sum_{p=1}^{\infty} p^2 U_p \sin p\pi\frac{l}{b}. \tag{20}$$

We simplify Eq. (20), using the fast Fourier series:

$$\sum_{p=1}^{\infty}\frac{(-1)^p}{p^3\pi^3}\sin p\pi\frac{x}{b} = \frac{x^3 - b^2 x}{12b^3}, \quad \sum_{p=1}^{\infty}\frac{1}{p^3\pi^3}\sin p\pi\frac{x}{b} = \frac{x(b-x)(2b-x)}{12b^3}, \quad 0 \le x \le b. \tag{21}$$

With the aid of the first series from (21) we can establish that the coefficient of $\sum_{p=1}^{\infty} U'_p \sin p\pi l/b$ in Eq. (20) is equal to zero. Therefore Eq. (20) will not contain the derivatives U'_m, and after simplifications with the aid of the second series from (21) it takes the form

$$a\frac{\pi^2}{b^2}\sum_{p=1}^{\infty} p^2 U_p \sin p\pi\frac{l}{b} = \frac{l\,(b-l)}{3a}\left(\mu_1'' - f_1^{*\prime}\right) + f(t,l) - f_1^*$$

$$+\left(l'\frac{3l^2-b^2}{l^3-b^2 l} + \frac{6a}{l^2-b^2}\right)A(t) + \frac{l}{b}\mu_1' - C(t) + \sum_{p=1}^{\infty} U_p p\pi\frac{l'}{b}\cos p\pi\frac{l}{b}$$

$$-\frac{2l}{b\,(l^2-b^2)}(ll'+3a)\,\mu_3 - \left(l'\frac{3l^2-b^2}{l^3-b^2 l} + \frac{6a}{l^2-b^2}\right)\sum_{p=1}^{\infty} U_p \sin p\pi\frac{l}{b}\,. \tag{22}$$

Resolving (22) for $U_1(t)$, we obtain

$$U_1 = \Bigg[-\mu_2' + \mu_1' - \frac{l'}{b}\mu_1 + \frac{l^2-lb}{6a}\left(\mu_1'' - f_1^{*\prime}\right) + \left(\mu_1' - f_1^{*\prime}\right)\frac{l}{l+b}\left(l'\frac{b-l}{2a} - 3\right) + f(t,l)$$

$$+\left(l'\frac{3l^2-b^2}{l^3-b^2 l} + \frac{6a}{l^2-b^2}\right)\left(\mu_2 - \mu_1 + \frac{l}{b}\mu_1\right) - f_1^* - \frac{2l}{b\,(l^2-b^2)}(ll'+3a)\,\mu_3$$

$$-a\frac{\pi^2}{b^2}\sum_{p=2}^{\infty} p^2 U_p \sin p\pi\frac{l}{b} - \left(l'\frac{3l^2-b^2}{l^3-b^2 l} + \frac{6a}{l^2-b^2}\right)\sum_{p=2}^{\infty} U_p \sin p\pi\frac{l}{b}$$

$$+\sum_{p=2}^{\infty} U_p p\pi\frac{l'}{b}\cos p\pi\frac{l}{b}\Bigg]\left\{\left(a\frac{\pi^2}{b^2} + l'\frac{3l^2-b^2}{l^3-b^2 l} + \frac{6a}{l^2-b^2}\right)\sin\pi\frac{l}{b} - \pi\frac{l'}{b}\cos\frac{l}{b}\right\}^{-1}. \tag{23}$$

Equation (23) is equivalent to one of the equations of system (19). Therefore in (19) we replace the first equation at $m = 1$ by Eq. (23) and we will assume that in (19) $m = 2-\infty$. Thus, now the closed system will consist of the equations: (17) for $\mu_3(t)$, (19) for $U_m(t)$ at $m = 2-\infty$, and (23) for $U_1(t)$ with initial conditions (16) for $\mu_3(0)$ and $U_m(0)$. After the elimination of $U_1(t)$ with the aid of (23), the system (17) and (19) can be resolved for the first derivatives $\mu_3'(t)$, $U_m'(t)$ at $m = 2-\infty$ and can be represented in the standard form $y' = s(t, y)$ suitable for computers.

NOTATION

a, thermal diffusivity, m^2/s; b, dimensions of the extended region in the form of a segment, m; $C^{(2)}$, $C^{(4)}$, spaces of differentiable functions; c_0, nondimensional parameter of an internal source; $f_1^*(t)$, $f_2^*(t)$, values of the source $f^*(t, x)$ at points $x = 0$ and $x = b$, respectively; f, the known right-hand side of heat-conduction equation (1) having the meaning of a source; $l(t)$, coordinate of the right mobile boundary; $l(0)$, nondimensional initial length of the segment; m, summation index; m_0, number of terms held in the Fourier series; t, time, s; t_0, time of consideration of the process, s; $U_m(t)$, $f_m(t)$, Fourier coefficients of functions $U(t, x)$, $f(t, x)$; x, Cartesian coordinate, m; $\mu_1(t)$, $\mu_2(t)$, $\mu_3(t)$, boundary conditions at $x = 0$, $x = l(t)$, $x = b$; $\mu_4(t)$, $\mu_5(t)$, second partial derivatives of temperature with respect to the coordinate x at the boundaries $x = 0$, $x = b$; $\varphi(x)$, $\varphi_\square(x)$, initial conditions in regions Ω, $\Omega_\square$; Ω, region with a mobile boundary; $\Omega_\square$, expanded immobile region; $\Gamma_\square$, its boundary. Subscripts and superscripts: m, number of terms in

Fourier series; □, quantities relating to the auxiliary problem in the expanded region $\Omega_{\square}$; 0, initial conditions or fixed values of t_0, m_0; t, partial derivative with respect to time; x, partial derivative with respect to coordinate; *, quantities relating to expansion of the region $\Omega_{\square}$; ′ , ″ , derivatives.

REFERENCES

1. A. N. Tikhonov and A. A. Samarskii, *Equations of Mathematical Physics* [in Russian], Nauka, Moscow (1977).
2. É. M. Kartashov, *Analytical Methods in the Theory of Thermal Conductivity of Solids* [in Russian], Vysshaya Shkola, Moscow (2001).
3. A. D. Chernyshov, Solution of a plane, axisymmetric, and three-dimensional single-phase Stefan problem, *Inzh.-Fiz. Zh.*, **27**, No. 2, 341–350 (1974).
4. A. D. Chernyshov, *Method of Angular Superpositions (Boundary-Value Problems of Mechanics and Applied Mathematics)* [in Russian], VGTA, Voronezh (2009).
5. V. A. Il'in, *Spectral Theory of Differential Operators. Self-Conjugate Differential Operators* [in Russian], Nauka, Moscow (1991).
6. G. P. Tolstov, *Fourier Series* [in Russian], Nauka, Moscow (1980).